

An Isobar Model of $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$

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Abstract

This paper presents several matrix elements for processes that may contribute to $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$ derived using the isobar model and the Lorentz invariant amplitude method. The formulas may be used to measure the electromagnetic form factors of the ρ meson in experimental data. Quantum Chromodynamics (QCD) calculations predict the electromagnetic form factors of the ρ meson which can be compared to the measured values. The formulas in this paper may be used to study the a_1 meson if $e^+e^- \rightarrow a_1\pi$ contributes to $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$.

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I. INTRODUCTION

The isobar model (described below) is a phenomenological theory of intermediate energy (a few GeV) strong interactions [1–8]. Phenomenology is “a descriptive or classificatory account of the phenomena of a given body of knowledge, without any further attempt at explanation” [9]. In this sense, the isobar model is phenomenology. Quantum Chromodynamics (**QCD**) hopefully provides a fundamental explanation of the interactions described by the isobar model. The isobar model cannot be derived at present from **QCD**. It presumably is not a fundamental explanation of strong interactions, but provides a description of many phenomena in strong interactions.

This paper derives formulas for the isobar model processes that may contribute to the reaction $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$. These formulas can be compared to $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$ data from experiments such as CLEO, CLEO II, the Beijing Electron Spectrometer (BES), **Mark III**, and **DM2**.

The formulas can be used to measure the electromagnetic form factors of the ρ meson in experimental data on $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$. There are at least two predictions of the electromagnetic form factors of the ρ based on **QCD** arguments to compare to the experimental measurements [10–13]. The isobar model formulas can be used to determine the properties of the a_1 resonance if $e^+e^- \rightarrow a_1\pi$ is a significant contributing process to $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$. A more detailed discussion of these results can be found in the thesis on which this paper is based [14].

II. THE ISOBAR MODEL

The isobar model assumes that particle production and decay proceeds via resonances. For example, in the process $\gamma p \rightarrow \pi^0 p$, pion photoproduction, the isobar model assumes the matrix element is of the general form

$$M = \sum_i A_i M E_i(\epsilon_\gamma, \epsilon_p) \frac{1}{m^2 - m_i^2 - i\Gamma_i m} \quad (1)$$

where m_i is the mass of resonance i and Γ_i is the width of the resonance. This is the Breit-Wigner formula which originated in studies of atomic and nuclear physics. The isobar model uses several variants of the Breit-Wigner formula. The Breit-Wigner formulas are the non-relativistic Breit-Wigner formula with constant mass m_0 and width Γ_0 , the relativistic Breit-Wigner formula with constant mass and width, and Breit-Wigner formulas where the width $\Gamma(m)$ is a function of the mass.

If the resonance is narrow, there is little difference between a Breit-Wigner formula with a constant width $\Gamma_0 = \Gamma(m_0)$ and a Breit-Wigner formula with a width that is a function of the mass. If the resonance is broad, the Breit-Wigner formula with mass-dependent width

$\Gamma(m)$ may be significantly different from a Breit-Wigner with constant width. Additional assumptions are required to specify the mass-dependent width $\Gamma(m)$.

$ME_i(\epsilon_\gamma, \epsilon_p)$ is a Lorentz invariant matrix element incorporating the spin and momentum dependence of the interaction. A_i in Equation 1 could be a complex function of the mass $m^2 = (p_\gamma + p_p)^2$ but it is generally a complex constant. This is certainly a reasonable approximation if the resonance i is narrow. If the resonance is broad, this approximation may be invalid.

Pion photoproduction is presumed to proceed by

$$\gamma p \rightarrow \sum_I \text{Resonance } I \rightarrow \pi^0 p$$

In the isobar model, the S matrix [15,16] is the sum of products of Breit-Wigner factors, each corresponding to a resonance, and complex matrix elements incorporating spin-dependence. Physically, this means the interaction proceeds by sequential two-body interactions. For example, $A + B + C \rightarrow D + E + F$ might proceed by $A + B \rightarrow D + X$ followed by $C + X \rightarrow E + F$. X is a resonance and appears as a term

$$\frac{1}{(p_A + p_B - p_C)^2 - M_X^2 + i\Gamma_X M_X}$$

The process might proceed by $A + C \rightarrow D + Y$ and $B + Y \rightarrow E + F$ as well, where Y is a different resonance.

For particle decay, the isobar model is equivalent to *sequential two-body decay* of resonances. For example, the decay $A \rightarrow B + C + D$ is actually $A \rightarrow X + D$ followed by $X \rightarrow B + C$. The intermediary resonance X appears as a Breit-Wigner term with mass m_X and width Γ_X in the S matrix.

The Breit-Wigner formula is an assumption of the isobar model. This formula originated of atomic and nuclear physics. Narrow peaks in cross-sections occur in atomic and nuclear physics. These are typically interpreted as bound states in a central potential. The Breit-Wigner formula was derived in this context by applying the mathematics of a forced damped oscillator such as a tuning fork to atomic and nuclear systems.

When resonances were discovered in pion-nucleon scattering, an analogy was made to the much narrower states observed in atomic and nuclear physics. As part of this analogy, the Breit-Wigner formula was applied to low-energy hadronic interactions.

The isobar model incorporates a Lorentz invariant matrix element to represent the angular distributions in the interaction due to the spin of the interacting particles. The evaluation of these matrix elements comprises the bulk of this paper.

The isobar model assumes that the resonances have the same properties — the same mass, the same width, the same charge, the same quantum numbers — in all processes. The mass and width refer to the parameters of the Breit-Wigner formula for the resonance. The isobar model assumes a fundamental spectrum of resonances with unique properties that describe all low-energy hadronic interactions. The same resonances occur in e^+e^- annihilation, pion-nucleon scattering, and other processes. In practice, this spectrum is finite because only enough energy is present to create the lower energy resonances.

Generally, isobar models assume that only resonances below a certain cut-off angular momentum contribute to low-energy nuclear processes. Resonances with a spin above 2 are usually ignored in isobar models. The higher spin resonances presumably have larger masses. This makes intuitive sense if the resonances are excited states of quarks in a potential. The energy to create the high spin resonances is not available. In practice, radial excitations are frequently ignored for the same reason.

In summary, the isobar model assumes:

- The S matrix should be a Lorentz invariant function of initial and final state 4-momenta and spins.
- The S matrix can be represented as the sum of products of Breit-Wigners which represent known resonances. Physically, this can be thought of as representing the process as a sequence of two-body interactions.
- Resonances are represented by the Breit-Wigner formula. Frequently, the relativistic Breit-Wigner formula with constant mass and width is used.

- The resonances are the same — have the same mass, width, charge, and other quantum numbers — in all processes: e^+e^- annihilation, pion-nucleon scattering, kaon-nucleon scattering, pion photoproduction, pion electroproduction, and so forth.
- Each two-body process has an associated Lorentz invariant matrix element that represents the angular distributions in the two-body process due to the spins of the interacting particles. The rules for constructing these matrix elements are described below. The helicity formalism is an alternate scheme for constructing these matrix elements that is not used in this paper.
- Each two-body process has an associated amplitude which is treated as a complex constant. For spinless particles, the most general Lorentz invariant form of this amplitude for a two-body process would be an arbitrary function of the two-body invariant mass in the two-body process. A complex constant is a reasonable approximation for narrow resonances. For broad resonances, it constitutes an *assumption*.

In practice, only resonances with less than a cut-off spin J are used and radial excitations are frequently ignored.

III. LORENTZ INVARIANT AMPLITUDE METHOD

Lorentz invariant matrix elements are constructed by describing a decay as sequential two-body or three-body decay vertices. In $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$, $\rho \rightarrow \pi\pi$ is a two-body vertex and $\omega \rightarrow \pi\pi\pi$ is a three-body vertex. The total matrix element is the product of the matrix elements for each vertex. These matrix elements are constructed from the 4-momenta and polarization 4-vectors of the in-going and out-going states.

In this paper, the following notation will be used. A 4-vector is represented as follows:

$$a_\mu = a = (a_x, a_y, a_z, a_t) \tag{2}$$

and

$$a^\mu = (-a_x, -a_y, -a_z, a_t) \quad (3)$$

A 4-momentum is:

$$p_\mu = p = (p_x, p_y, p_z, E) \quad (4)$$

where E is the energy.

This analysis uses two kinds of 4-vectors. 4-momenta change sign under the parity transformation. Polarization 4-vectors, which are usually represented by ϵ , do not change sign under the parity transformation. The amplitudes formed by the Lorentz invariant amplitude formalism must have definite parity properties, determined by the intrinsic parities of the particles as described below.

The Minkowski metric $g_{\mu\nu}$ is defined as:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5)$$

Then, the dot product of two 4-vectors can be represented as:

$$a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu \quad (6)$$

$$a_\mu b^\mu = \delta_\nu^\mu a_\mu b^\nu \quad (7)$$

$$a_\mu b^\mu = (a \cdot b) \quad (8)$$

$$(a \cdot b) = a_t b_t - a_x b_x - a_y b_y - a_z b_z \quad (9)$$

δ_ν^μ is the Kronecker delta.

Notice, in this notation:

$$p \cdot p = M^2 \quad (10)$$

With the other possible choice of metric,

$$p \cdot p = -M^2 \quad (11)$$

The polarization 4-vectors $\epsilon(\lambda)$ take the following form in this notation:

For a massive spin one (1) particle with helicity λ with momentum \bar{p} along the x axis,

$$\epsilon(\lambda = \pm 1) = \mp(0, 1, \pm i, 0)/\sqrt{2} \quad (12)$$

$$\epsilon(\lambda = 0) = (p, 0, 0, E)/M \quad (13)$$

where M is mass of particle. or

$$\epsilon(\lambda = 0) = (p, 0, 0, E)/|p| \quad (14)$$

For narrow resonances, Equation 13 and Equation 14 are equivalent. However for Equation 14, note that if the massive vector is broad, then $p \cdot p \neq M^2$ in general. Equation 14 is the correct formula to use in general. Equation 13 is an approximation. These polarization 4-vectors are orthonormal. They satisfy:

$$\epsilon(\lambda_i)\epsilon(\lambda_j)^* = \delta_{ij} \quad (15)$$

Notice that one dots with complex conjugate to get this.

For a massless vector particle (such as a photon):

$$\epsilon \cdot p = \epsilon_\mu p^\mu = 0 \quad (16)$$

The Lorentz invariant amplitudes are also built using the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$. The Levi-Civita tensor is defined by

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} -1, & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of } xyzt; \\ +1, & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of } xyzt; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\epsilon^{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta}$$

The Levi-Civita tensor changes sign under the parity transformation. It has some other properties which will be discussed in the section below on the $\omega\pi$ model.

The Levi-Civita tensor enters into the amplitudes in forms

$$\epsilon_{\mu\nu\rho\sigma}A^\mu B^\nu C^\rho D^\sigma$$

The Levi-Civita tensor is a determinant or a signed volume of a box defined by four 4-vectors. If a 4-vector is repeated in the above form (e.g. $A = B$) then the form is 0.

The matrix elements are built out of the 4-momenta, polarization 4-vectors, Breit-Wigners, and the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$. The polarization of the intermediate states must be summed over using the completeness relation.

$$\sum_{\lambda} \epsilon_{\mu}^{\lambda*} \epsilon_{\nu}^{\lambda} = -g_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{p^2} \quad (17)$$

where ϵ is the polarization 4-vector of the intermediate state, p_{μ} is the 4-momentum of the intermediate state, and M is the pole-mass of the intermediate state. $g_{\mu\nu}$ is the Minkowski tensor. For a narrow resonance, p^2 can be replaced by M^2 in the completeness relation. This analysis involves the ρ which is broad.

In practice, the completeness relation appears in forms of the type

$$\sum_{\lambda} (A \cdot \epsilon(\lambda))(B \cdot \epsilon(\lambda)) = \left[-(A \cdot B) + \frac{(A \cdot P_s)(P_s \cdot B)}{P_s^2} \right] \quad (18)$$

where A and B are 4-vectors, either four momenta or polarization 4-vectors. $\epsilon(\lambda)$ is the polarization of an intermediate state which is summed over. P_s is the momentum of the intermediate state s .

The completeness relation also appears in forms of the type

$$\begin{aligned} \sum_{\lambda_a \lambda_b} (A \cdot \epsilon_a(\lambda_a))(\epsilon_a(\lambda_a) \cdot \epsilon_b(\lambda_b))(\epsilon_b(\lambda_b) \cdot B) = & - \left[-(A \cdot B) + \frac{(A \cdot P_b)(P_b \cdot B)}{P_b^2} \right] \\ & + \frac{P_b \cdot B}{P_b^2} \left[-(A \cdot B) + \frac{(A \cdot P_a)(P_a \cdot B)}{P_a^2} \right] \end{aligned} \quad (19)$$

The Lorentz invariant amplitudes usually reduce to formula constructed of 4-vector products using the expansions in Equations 18 and 19 above. This allows one to drop the Greek

indices and provides a form for the amplitudes which is easy to compute with a 4-vector dot product function when implementing the model on a computer.

The specific rules for forming amplitudes (or matrix elements) are:

- The amplitudes must be formed by contracting combinations of the available 4-momenta and polarization 4-vectors. The amplitudes must be Lorentz invariant. 4-vectors are not Lorentz invariant, but the contractions are Lorentz invariants. All of the polarization 4-vectors *must* be used to form the amplitudes. The 4-momenta may or may not be used.
- Different amplitudes for the same process must be linearly independent to within factors which are rotationally invariant. These factors are envisioned as slowly varying functions of invariant masses.
- The amplitudes must be either scalars or pseudoscalars depending on the required parity properties. The product of the intrinsic parities of the decaying particle and the decay products determines the required parity transformation properties of the amplitude.
- Photons must satisfy $\epsilon^\mu p_\mu = 0$. ϵ is the polarization 4-vector of the photon and p is the 4-momentum of the photon.
- Intermediate states must be summed over helicities using the completeness relation above.
- Intermediate states should include a mass dependent phase. This is the Breit-Wigner form.
- If the process being modeled is an eigenstate of $C(G)$ then the amplitude must be appropriately symmetrized so that the final state it describes is also an eigenstate of $C(G)$. C is charge conjugation which transforms a particle into its anti-particle. G refers to G parity.

A. Mass Dependent Widths

The Particle Data Book and other sources only give a single width at the pole mass $\Gamma_0 = \Gamma(m_0)$. As mentioned above, the Breit-Wigner form contains a mass dependent width. For a particle d which decays to N particles, the width is determined by

$$\Gamma(d \rightarrow 1, \dots, n) = \frac{1}{2m} \int dLIPS(m^2, P_1, \dots, P_n) |ME|^2 \quad (20)$$

where ME is the matrix element of the decay. $dLIPS$ refers to integration of Lorentz Invariant Phase Space.

This paper is concerned with two decays: $\rho \rightarrow \pi\pi$ and $\omega \rightarrow \pi\pi\pi$. The volume of phase space changes as the invariant mass of the decaying particle d changes. Thus, for this reason alone, the width is mass dependent. The isobar model needs formulas that relate $\Gamma(m)$ to the width Γ_0 and the mass m of the intermediate state.

For massive spin one (vector) particles decaying to two particles, the formula

$$\Gamma(m) = \Gamma_0 \frac{m_0}{m} \left(\frac{p_{cm}}{p_{cm0}} \right)^3 \frac{1 + (Rp_{cm0})^2}{1 + (Rp_{cm})^2} \quad (21)$$

can be used. It is used for the ρ resonances. However, it is inappropriate for the ω which decays to three pions. In this instance, the $\Gamma(m)$ can be calculated by explicitly integrating the matrix element squared using the formula for the mass dependent width above.

In this, $\sqrt{\frac{1+(Rp_{cm0})^2}{1+(Rp_{cm})^2}}$ are the Blatt-Weisskopf barrier penetration factors [17]. The Blatt-Weisskopf barrier penetration factors represent mathematically that the particles are not point particles. R is an estimate of the radius of a strongly interacting particle. Notice that if $R = 0$ then the penetration factors are identically 1. The width Γ is the integral of the square of the matrix element of the decay. The Blatt-Weisskopf factor $\sqrt{\frac{1+(Rp_{cm0})^2}{1+(Rp_{cm})^2}}$ appears in the matrix element. This relationship between the width and the matrix element is why the square of the Blatt-Weisskopf barrier penetration factor appears in the formula for the width. p_{cm} is the momentum of the two-body decay in the resonance rest frame. The value of R is not definite. The value 1 fermi or 5 GeV^{-1} is usually used.

IV. LORENTZ INVARIANT AMPLITUDE MODELS FOR $E^+E^- \rightarrow \pi^+\pi^-\pi^0\pi^0$

This section contains the main results of this paper, Lorentz Invariant Amplitudes for several processes that may contribute to $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$. This paper envisions the process $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$ as the superposition of several modes listed in Table I. All of the modes except for the last are represented mathematically as Lorentz invariant complex amplitudes that interfere.

The last mode in Table I is non-resonant four pions, usually production of $\pi^+\pi^-\pi^0\pi^0$ according to a four-body phase space distribution added incoherently to the other modes. Technically, the isobar model precludes this last mode. It is incorporated in isobar model analyses as a substitute for unmodelled modes, detector simulation errors, and so forth when performing fits to experimental data.

A. The $e^+e^- \rightarrow \rho^+\rho^-$ Mode

This is $V \rightarrow VV$ process where V indicates a vector particle. There are three polarization 4-vectors ϵ_{γ^*} , ϵ_{ρ^+} , and ϵ_{ρ^-} . ϵ_{γ^*} is the polarization of the virtual photon formed when the electron and positron annihilate. There are two independent 4-momenta P and Q . $P = (0, 0, 0, E_{cm}) = P_{\rho^+} + P_{\rho^-}$ and $Q = P_{\rho^+} - P_{\rho^-}$. There are two neutral pions π^0 . These are referred to in this analysis as π_1^0 for the more energetic pion and π_2^0 for the less energetic pion. There are two ways to form the 4-momenta of the two ρ mesons: $P_{\rho^+} = P_{\pi^+} + P_{\pi_1^0}$ and $P_{\rho^-} = P_{\pi^-} + P_{\pi_2^0}$ or $P_{\rho^+} = P_{\pi^+} + P_{\pi_2^0}$ and $P_{\rho^-} = P_{\pi^-} + P_{\pi_1^0}$. It will be necessary to add the matrix elements for these two cases to insure that the final amplitude has the correct Bose symmetry. As a first step, the decay of the ρ 's is ignored.

There are three independent amplitudes that can be constructed according to the rules of the Lorentz invariant formalism. These are:

$$ME_1 = (\epsilon_{\gamma^*} \cdot Q)(\epsilon_{\rho^+} \cdot \epsilon_{\rho^-}) \quad (22)$$

$$ME_2 = (\epsilon_{\gamma^*} \cdot Q)(\epsilon_{\rho^+} \cdot Q)(\epsilon_{\rho^-} \cdot Q) \quad (23)$$

$$ME_3 = (\epsilon_{\gamma^*} \cdot \epsilon_{\rho^+})(\epsilon_{\rho^-} \cdot Q) + (\epsilon_{\gamma^*} \cdot \epsilon_{\rho^-})(\epsilon_{\rho^+} \cdot Q) \quad (24)$$

Note that these are the matrix elements at the $\gamma^* \rightarrow \rho^+ \rho^-$ vertex only. The full formulas (see below) must include the $\rho \rightarrow \pi\pi$ decay. All of these formulas are parity – and charge conjugation – eigenstates. The $(\rho^+ \rho^-)$ final state is a charge conjugation eigenstate [18]. Kramer and Walsh’s paper quotes these formulas using a different notation. Their paper does not consider the ρ decay.

Thus, the total matrix element would be

$$ME(e^+e^- \rightarrow \rho^+ \rho^-) = A_1 ME_1 + A_2 ME_2 + A_3 ME_3 \quad (25)$$

where A_i are complex parameters. The parameters correspond to the electric, magnetic, and electric quadrupole form factors of the ρ . The relations between the form factors F_E , F_M , and F_Q and the A_i amplitudes is

$$A_1 = F_E + \frac{2}{3}F_Q \quad (26)$$

$$A_2 = (2(1 - \eta))^{-1}[(1 - \frac{2}{3}\eta)F_Q - F_E + F_M] \quad (27)$$

$$A_3 = F_M \quad (28)$$

where

$$\eta = \frac{p^2}{4m_\rho} \quad (29)$$

p being $(0, 0, 0, E_{cm})$

B. Incorporating ρ Decay in Amplitudes

So far, the propagation and decay of the ρ 's has been neglected. This adds a Breit-Wigner factor for each ρ and a factor $\epsilon_\rho \cdot Q_\rho$. $Q_{\rho^\pm} = P_{\pi^\pm} - P_{\pi^0}$ where the pions are the decay products of the ρ meson. In addition, the polarization of the intermediate ρ must be summed over using the completeness relation.

Thus, the matrix elements become

$$ME_1 = \sum_{\lambda_{\rho^+}\lambda_{\rho^-}} (\epsilon_{\gamma^*} \cdot Q)(\epsilon_{\rho^+} \cdot \epsilon_{\rho^-})(\epsilon_{\rho^+} \cdot Q_{\rho^+})(\epsilon_{\rho^-} \cdot Q_{\rho^+}) \frac{1}{(m_{\rho^+}^2 - m_0^2 - im_{\rho^+}\Gamma_{\rho^+})} \frac{1}{(m_{\rho^-}^2 - m_0^2 - im_{\rho^-}\Gamma_{\rho^-})} \quad (30)$$

Now perform the summation over the polarizations of the ρ^+ and ρ^- . Recall the relations

$$\sum_{\lambda} (A \cdot \epsilon(\lambda))(B \cdot \epsilon(\lambda)) = -(A \cdot B) + \frac{(A \cdot P)(B \cdot P)}{M^2} \quad (31)$$

$$\begin{aligned} \sum_{\lambda_a\lambda_b} (A \cdot \epsilon_a)(\epsilon_a \cdot \epsilon_b)(\epsilon_b \cdot B) &= \left[-(A \cdot B) + \frac{(A \cdot P_b)(P_b \cdot B)}{P_b^2} \right] \\ &\quad - \frac{(A \cdot P_a)}{P_a^2} \left(-(P_a \cdot B) + \frac{(P_a \cdot P_b)(P_b \cdot B)}{P_b^2} \right) \end{aligned} \quad (32)$$

Using these relations, the full matrix elements with the ρ decay become:

$$\begin{aligned} ME_1 &= (\epsilon_{\gamma^*} \cdot Q) \left\{ \left[-(Q_{\rho^+} \cdot Q_{\rho^-}) + \frac{(Q_{\rho^+} \cdot P_{\rho^-})(P_{\rho^-} \cdot Q_{\rho^-})}{P_{\rho^-}^2} \right] \right. \\ &\quad \left. - \frac{(Q_{\rho^+} \cdot P_{\rho^+})}{P_{\rho^+}^2} \left(-(P_{\rho^+} \cdot Q_{\rho^-}) + \frac{(P_{\rho^+} \cdot P_{\rho^-})(P_{\rho^-} \cdot Q_{\rho^-})}{P_{\rho^-}^2} \right) \right\} \\ &\quad \frac{1}{(m_{\rho^+}^2 - m_0^2 - im_{\rho^+}\Gamma_{\rho^+})} \frac{1}{(m_{\rho^-}^2 - m_0^2 - im_{\rho^-}\Gamma_{\rho^-})} \end{aligned} \quad (33)$$

$$\begin{aligned} ME_2 &= (\epsilon_{\gamma^*} \cdot Q) \left(\left[-(Q \cdot Q_{\rho^+}) + \frac{(Q \cdot P_{\rho^+})(P_{\rho^+} \cdot Q_{\rho^+})}{P_{\rho^+}^2} \right] \right. \\ &\quad \left. \left(\left[-(Q \cdot Q_{\rho^-}) + \frac{(Q \cdot P_{\rho^-})(P_{\rho^-} \cdot Q_{\rho^-})}{P_{\rho^-}^2} \right] \right) \right) \\ &\quad \frac{1}{(m_{\rho^+}^2 - m_0^2 - im_{\rho^+}\Gamma_{\rho^+})} \frac{1}{(m_{\rho^-}^2 - m_0^2 - im_{\rho^-}\Gamma_{\rho^-})} \end{aligned} \quad (34)$$

$$\begin{aligned} ME_3 &= \left(\left[-(\epsilon_{\gamma^*} \cdot Q_{\rho^+}) + \frac{(\epsilon_{\gamma^*} \cdot P_{\rho^+})(P_{\rho^+} \cdot Q_{\rho^+})}{P_{\rho^+}^2} \right] \right) \\ &\quad \left(\left[-(Q \cdot Q_{\rho^-}) + \frac{(Q \cdot P_{\rho^-})(P_{\rho^-} \cdot Q_{\rho^-})}{P_{\rho^-}^2} \right] \right) \\ &\quad + \left(\left[-(\epsilon_{\gamma^*} \cdot Q_{\rho^-}) + \frac{(\epsilon_{\gamma^*} \cdot P_{\rho^-})(P_{\rho^-} \cdot Q_{\rho^-})}{P_{\rho^-}^2} \right] \right) \\ &\quad \left(\left[-(Q \cdot Q_{\rho^+}) + \frac{(Q \cdot P_{\rho^+})(P_{\rho^+} \cdot Q_{\rho^+})}{P_{\rho^+}^2} \right] \right) \\ &\quad \frac{1}{(m_{\rho^+}^2 - m_0^2 - im_{\rho^+}\Gamma_{\rho^+})} \frac{1}{(m_{\rho^-}^2 - m_0^2 - im_{\rho^-}\Gamma_{\rho^-})} \end{aligned} \quad (35)$$

using these relations, the final expression for the amplitude for the $\rho^+\rho^-$ mode(s) becomes

$$\begin{aligned}
\Psi_{\rho^+\rho^-}(\Omega) = & R_1 e^{i\phi_1} (ME_1(\pi_1^0\pi_2^0) + ME_1(\pi_2^0\pi_1^0)) \\
& + R_2 e^{i\phi_2} (ME_2(\pi_1^0\pi_2^0) + ME_2(\pi_2^0\pi_1^0)) \\
& + R_3 e^{i\phi_3} (ME_3(\pi_1^0\pi_2^0) + ME_3(\pi_2^0\pi_1^0))
\end{aligned} \tag{36}$$

$ME_i(\pi_1^0\pi_2^0)$ and $ME_i(\pi_2^0\pi_1^0)$ refer to the two choices of π^0 to form the 4-vectors in order to satisfy Bose statistics under interchange of the two identical neutral pions. Defining π_1^0 as the more energetic neutral pion (e.g. $E_{\pi_1^0} \geq E_{\pi_2^0}$) in the center of mass of the four pions frame of reference provides a well-defined and unique way to label the two neutral pions in real data.

C. Simplification of $\rho^+\rho^-$ Amplitudes

The preceding amplitudes are complicated. However, only a few terms dominate the amplitudes. The factors $(Q_{\rho^\pm} \cdot P_{\rho^\pm})$ in the formulas above are the reason for this.

$$(Q_{\rho^\pm} \cdot P_{\rho^\pm}) = (P_{\pi^\pm} - P_{\pi^0}) \cdot (P_{\pi^\pm} + P_{\pi^0}) \tag{37}$$

This becomes

$$(Q_{\rho^\pm} \cdot P_{\rho^\pm}) = (P_{\pi^\pm} \cdot P_{\pi^\pm}) - (P_{\pi^0} \cdot P_{\pi^0}) \tag{38}$$

which becomes

$$(Q_{\rho^\pm} \cdot P_{\rho^\pm}) = m_{\pi^\pm}^2 - m_{\pi^0}^2 \tag{39}$$

If the masses of the charged and neutral pions were the same, this term would be exactly zero. In this case the $\rho^+\rho^-$ amplitudes reduce to

$$\begin{aligned}
ME_1 = & -(\epsilon_{\gamma^*} \cdot Q)(Q_{\rho^+} \cdot Q_{\rho^-}) \\
& \frac{1}{(m_{\rho^+}^2 - m_0^2 - im_{\rho^+}\Gamma_{\rho^+})} \frac{1}{(m_{\rho^-}^2 - m_0^2 - im_{\rho^-}\Gamma_{\rho^-})}
\end{aligned} \tag{40}$$

$$\begin{aligned}
ME_2 = & (\epsilon_{\gamma^*} \cdot Q) \left(- \left((Q \cdot Q_{\rho^+}) + (Q \cdot Q_{\rho^-}) \right) \right) \\
& \frac{1}{(m_{\rho^+}^2 - m_0^2 - im_{\rho^+}\Gamma_{\rho^+})} \frac{1}{(m_{\rho^-}^2 - m_0^2 - im_{\rho^-}\Gamma_{\rho^-})}
\end{aligned} \tag{41}$$

$$ME_3 = - \left((\epsilon_{\gamma^*} \cdot Q_{\rho^+})(Q \cdot Q_{\rho^-}) + (\epsilon_{\gamma^*} \cdot Q_{\rho^-})(Q \cdot Q_{\rho^+}) \right) \frac{1}{(m_{\rho^+}^2 - m_0^2 - im_{\rho^+} \Gamma_{\rho^+})} \frac{1}{(m_{\rho^-}^2 - m_0^2 - im_{\rho^-} \Gamma_{\rho^-})} \quad (42)$$

Although the masses of the charged and neutral pions are slightly different, the terms above dominate the $\rho^+\rho^-$ amplitudes. Another consequence is that the $\rho^+\rho^-$ amplitudes are largely insensitive to the choice of $\frac{1}{M^2}$ or $\frac{1}{P^2}$ in the completeness relation.

D. The $e^+e^- \rightarrow a_1\pi$ Mode

This process consists of

$$e^+e^- \rightarrow a_1^\pm \pi^\mp$$

where

$$a_1^\pm \rightarrow \rho^\pm \pi^0$$

and

$$\rho^\pm \rightarrow \pi^\pm \pi^0$$

At the $e^+e^- \rightarrow \gamma^* \rightarrow a_1\pi$ step the relevant 4-vectors are ϵ_γ , ϵ_{a_1} , $P = P_{a_1} + P_\pi$ and $Q = P_{a_1} - P_\pi$. There are two independent amplitudes:

$$ME_{e^+e^- \rightarrow a_1\pi}^I = (\epsilon_{\gamma^*} \cdot \epsilon_{a_1}) \quad (43)$$

$$ME_{e^+e^- \rightarrow a_1\pi}^{II} = (\epsilon_{\gamma^*} \cdot Q)(\epsilon_{a_1} \cdot P) \quad (44)$$

At the a_1 vertex $a_1 \rightarrow \rho\pi$ there are four relevant 4-vectors: ϵ_{a_1} , ϵ_ρ , P_{a_1} , and $Q_{a_1} = P_\rho - P_\pi$. These are formed into two independent amplitudes:

$$ME_{a_1 \rightarrow \rho\pi}^I = (\epsilon_{a_1} \cdot \epsilon_\rho) \quad (45)$$

$$ME_{a_1 \rightarrow \rho\pi}^{II} = (\epsilon_{a_1} \cdot Q_{a_1})(\epsilon_\rho \cdot P_{a_1}) \quad (46)$$

At the ρ vertex, there is only one matrix element

$$ME_{\rho \rightarrow \pi\pi} = (\epsilon_\rho \cdot Q_\rho) \quad (47)$$

as seen before.

Combining all of these yields four independent amplitudes for the $a_1\pi$ process. These are

$$ME_{a_1\pi}^I = \sum_{\lambda_{a_1}\lambda_\rho} (\epsilon_{\gamma^*} \cdot \epsilon_{a_1})(\epsilon_{a_1} \cdot \epsilon_\rho)(\epsilon_\rho \cdot Q_\rho) \quad (48)$$

$$ME_{a_1\pi}^{II} = \sum_{\lambda_{a_1}\lambda_\rho} (\epsilon_{\gamma^*} \cdot \epsilon_{a_1})(\epsilon_{a_1} \cdot Q_{a_1})(\epsilon_\rho \cdot P_{a_1})(\epsilon_\rho \cdot Q_\rho) \quad (49)$$

$$ME_{a_1\pi}^{III} = \sum_{\lambda_{a_1}\lambda_\rho} (\epsilon_{\gamma^*} \cdot Q)(\epsilon_{a_1} \cdot P)(\epsilon_{a_1} \cdot \epsilon_\rho)(\epsilon_\rho \cdot Q_\rho) \quad (50)$$

$$ME_{a_1\pi}^{IV} = \sum_{\lambda_{a_1}\lambda_\rho} (\epsilon_{\gamma^*} \cdot Q)(\epsilon_{a_1} \cdot P)(\epsilon_{a_1} \cdot Q_{a_1})(\epsilon_\rho \cdot P_{a_1})(\epsilon_\rho \cdot Q_\rho) \quad (51)$$

Now, performing the sums over the polarizations yields:

$$ME_{a_1\pi}^I = \sum_{\lambda_\rho} \left[-(\epsilon_{\gamma^*} \cdot \epsilon_\rho) + \frac{(\epsilon_{\gamma^*} \cdot P_{a_1})(P_{a_1} \cdot \epsilon_\rho)}{P_{a_1}^2} \right] (\epsilon_\rho \cdot Q_\rho) \quad (52)$$

$$ME_{a_1\pi}^{II} = \left[-(\epsilon_{\gamma^*} \cdot Q_{a_1}) + \frac{(\epsilon_{\gamma^*} \cdot P_{a_1})(P_{a_1} \cdot Q_{a_1})}{P_{a_1}^2} \right] \left[-(P_{a_1} \cdot Q_\rho) + \frac{(P_{a_1} \cdot P_\rho)(P_\rho \cdot Q_\rho)}{P_\rho^2} \right] \quad (53)$$

$$ME_{a_1\pi}^{III} = \sum_{\lambda_\rho} (\epsilon_{\gamma^*} \cdot Q) \left[-(P \cdot \epsilon_\rho) + \frac{(P \cdot P_{a_1})(P_{a_1} \cdot \epsilon_\rho)}{P_{a_1}^2} \right] (\epsilon_\rho \cdot Q_\rho) \quad (54)$$

$$ME_{a_1\pi}^{IV} = (\epsilon_{\gamma^*} \cdot Q) \left[-(P \cdot Q_{a_1}) + \frac{(P \cdot P_{a_1})(P_{a_1} \cdot Q_{a_1})}{P_{a_1}^2} \right] \left[-(P_{a_1} \cdot Q_\rho) + \frac{(P_{a_1} \cdot P_\rho)(P_\rho \cdot Q_\rho)}{P_\rho^2} \right] \quad (55)$$

Performing the sums over the polarizations of the ρ deferred in two of the four yields:

$$ME_{a_1\pi}^I = - \left[-(\epsilon_{\gamma^*} \cdot Q_\rho) + \frac{(\epsilon_{\gamma^*} \cdot P_\rho)(P_\rho \cdot Q_\rho)}{P_\rho^2} \right] + \frac{\epsilon_{\gamma^*} \cdot P_{a_1}}{P_{a_1}^2} \left[-(P_{a_1} \cdot Q_\rho) + \frac{(P_{a_1} \cdot P_\rho)(P_\rho \cdot Q_\rho)}{P_\rho^2} \right] \quad (56)$$

$$ME_{a_1\pi}^{II} = \left[-(\epsilon_{\gamma^*} \cdot Q_{a_1}) + \frac{(\epsilon_{\gamma^*} \cdot P_{a_1})(P_{a_1} \cdot Q_{a_1})}{P_{a_1}^2} \right] \left[-(P_{a_1} \cdot Q_\rho) + \frac{(P_{a_1} \cdot P_\rho)(P_\rho \cdot Q_\rho)}{P_\rho^2} \right] \quad (57)$$

$$ME_{a_1\pi}^{III} = (\epsilon_{\gamma^*} \cdot Q) \left[- \left[-(P \cdot Q_\rho) + \frac{(P \cdot P_\rho)(P_\rho \cdot Q_\rho)}{P_\rho^2} \right] + \frac{P \cdot P_{a_1}}{P_{a_1}^2} \left[-(P_{a_1} \cdot Q_\rho) + \frac{(P_{a_1} \cdot P_\rho)(P_\rho \cdot Q_\rho)}{P_\rho^2} \right] \right] \quad (58)$$

$$ME_{a_1\pi}^{IV} = (\epsilon_{\gamma^*} \cdot Q) \left[-(P \cdot Q_{a_1}) + \frac{(P \cdot P_{a_1})(P_{a_1} \cdot Q_{a_1})}{P_{a_1}^2} \right] \left[-(P_{a_1} \cdot Q_\rho) + \frac{(P_{a_1} \cdot P_\rho)(P_\rho \cdot Q_\rho)}{P_\rho^2} \right] \quad (59)$$

These matrix elements must be multiplied by Breit-Wigner formulas for a_1 and ρ and Bose symmetrized.

$$\begin{aligned} \Psi_{a_1\pi}(\Omega) = & R_1 e^{i\phi_1} (ME_{a_1}^I(\pi_1^0 \pi_2^0) + ME_{a_1\pi}^I(\pi_2^0 \pi_1^0)) \\ & + R_2 e^{i\phi_2} (ME_{a_1}^{II}(\pi_1^0 \pi_2^0) + ME_{a_1\pi}^{II}(\pi_2^0 \pi_1^0)) \\ & + R_3 e^{i\phi_3} (ME_{a_1}^{III}(\pi_1^0 \pi_2^0) + ME_{a_1\pi}^{III}(\pi_2^0 \pi_1^0)) \\ & + R_4 e^{i\phi_4} (ME_{a_1}^{IV}(\pi_1^0 \pi_2^0) + ME_{a_1\pi}^{IV}(\pi_2^0 \pi_1^0)) \end{aligned} \quad (60)$$

E. The $e^+e^- \rightarrow \omega^0\pi^0$ Mode

This formalism can also provide a formula for the process

$$e^+e^- \rightarrow \omega^0\pi^0$$

$$\omega^0 \rightarrow \pi^+\pi^-\pi^0$$

The $e^+e^- \rightarrow \omega\pi$ vertex has the matrix element:

$$ME_{\omega\pi} = \epsilon_{\mu\nu\rho\sigma} \epsilon_{\gamma^*}^\mu \epsilon_\omega^\nu P^\rho Q^\sigma \quad (61)$$

where $P = P_\omega + P_{\pi^0}$ and $Q = P_\omega - P_{\pi^0}$

The $\omega \rightarrow \pi^+\pi^-\pi^0$ vertex is governed by the matrix element.

$$ME_{\pi\pi\pi} = \epsilon_{\mu\nu\rho\sigma} \epsilon_\omega^\mu P_{\pi^+}^\nu P_{\pi^-}^\rho P_\omega^\sigma \quad (62)$$

Since $P_\omega = P_{\pi^+} + P_{\pi^-} + P_{\pi^0}$, this formula is equivalent to

$$ME_{\pi\pi\pi} = \epsilon_{\mu\nu\rho\sigma} \epsilon_\omega^\mu P_{\pi^+}^\nu P_{\pi^-}^\rho P_{\pi^0}^\sigma \quad (63)$$

The complete matrix element is

$$ME_{\omega\pi} = \sum_{\lambda_\omega} \epsilon_{\mu\nu\rho\sigma} \epsilon_{\gamma^*}^\mu \epsilon_\omega^\nu P^\rho Q^\sigma \quad (64)$$

$$\epsilon_{\alpha\beta\gamma\delta} \epsilon_\omega^\alpha P_{\pi^+}^\beta P_{\pi^-}^\gamma P_\omega^\delta \frac{1}{(m_\omega^2 - m_0^2 - im_\omega \Gamma_\omega)} \quad (65)$$

This formula reduces to a simple formula in terms of dot products of 4-vectors (polarization 4-vectors and 4-momenta).

First, use the completeness relation for the ω to get

$$ME_{\omega\pi} = \epsilon^{\mu\nu\rho\sigma} P_\mu Q_\nu \epsilon_\rho^{\gamma^*} \epsilon^{\alpha\beta\gamma\delta} P_\beta^\omega P_\gamma^{\pi^+} P_\delta^{\pi^-} \left(-g_{\sigma\alpha} + \frac{P_\sigma^\omega P_\alpha^\omega}{M_\omega^2} \right) \frac{1}{(m_\omega^2 - m_0^2 - im_\omega \Gamma_\omega)} \quad (66)$$

Note that:

$$\frac{\epsilon^{\alpha\beta\gamma\delta} P_\beta^\omega P_\gamma^{\pi^+} P_\delta^{\pi^-} P_\sigma^\omega P_\rho^\omega}{M_\omega^2} = 0 \quad (67)$$

since P^ω is repeated. Thus the matrix element becomes the simpler form:

$$ME_{\omega\pi} = \epsilon^{\mu\nu\rho\sigma} P_\mu Q_\nu \epsilon_\rho^{\gamma^*} \epsilon^{\alpha\beta\gamma\delta} P_\beta^\omega P_\gamma^{\pi^+} P_\delta^{\pi^-} (-g_{\sigma\alpha}) \frac{1}{(m_\omega^2 - m_0^2 - im_\omega \Gamma_\omega)} \quad (68)$$

At this point, use the relation

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\rho\beta\gamma\delta} = \epsilon^{\rho\mu\nu\sigma} \epsilon_{\rho\beta\gamma\delta} = -\delta_{\beta\gamma\delta}^{\mu\nu\sigma} \quad (69)$$

where

$$\delta_{def}^{abc} = \begin{cases} 0 & \text{if index repeated in } abc \text{ or } def \\ 1 & \text{if } def \text{ is even permutation of } abc \\ -1 & \text{if } def \text{ is odd permutation of } abc \end{cases} \quad (70)$$

This is the Kronecker 3-delta. It is a generalization of the Kronecker delta δ_j^i .

$$\delta_b^a = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} \quad (71)$$

The Kronecker 2-delta is defined by

$$\delta_{cd}^{ab} = \begin{cases} 1 & \text{if } ab \text{ is even permutation of } cd \\ -1 & \text{if } ab \text{ is odd permutation of } cd \\ 0 & \text{if index repeated in } ab \text{ or } cd \end{cases} \quad (72)$$

The Kronecker 2-delta can be expressed in terms of the Kronecker delta:

$$\delta_{cd}^{ab} = (\delta_c^a \delta_d^b - \delta_d^a \delta_c^b) \quad (73)$$

The Kronecker 3-delta can be expanded in terms of the simple Kronecker delta as well:

$$\delta_{def}^{abc} = (\delta_d^a \delta_e^b \delta_f^c - \delta_e^a \delta_d^b \delta_f^c - \delta_d^a \delta_f^b \delta_e^c - \delta_f^a \delta_e^b \delta_d^c + \delta_e^a \delta_f^b \delta_d^c + \delta_f^a \delta_d^b \delta_e^c) \quad (74)$$

This leads to an expansion in terms of dot products:

$$\begin{aligned} A_\mu B_\nu C_\rho \delta_{\mu\nu\rho}^{\alpha\beta\gamma} D^\alpha E^\beta F^\gamma = & (A \cdot D)(B \cdot E)(C \cdot F) - (B \cdot D)(E \cdot A)(C \cdot F) \\ & - (A \cdot D)(C \cdot E)(B \cdot F) - (C \cdot D)(B \cdot E)(A \cdot F) \\ & + (B \cdot D)(C \cdot E)(A \cdot F) + (C \cdot D)(A \cdot E)(B \cdot F) \end{aligned} \quad (75)$$

This can be used to produce a form for the $\omega\pi$ matrix element in terms of dot products of the 4-momenta and the virtual photon polarization 4-vector.

$$\begin{aligned} ME = \epsilon_{\gamma^*} \cdot [& (P \cdot P_{\pi^-})(Q \cdot P_\omega)P_{\pi^+} \\ & - (P \cdot P_\omega)(Q \cdot P_{\pi^-})P_{\pi^+} \\ & - (P \cdot P_{\pi^+})(Q \cdot P_\omega)P_{\pi^-} \\ & + (P_\omega \cdot P)(Q \cdot P_{\pi^+})P_{\pi^-} \\ & + (P \cdot P_{\pi^+})(Q \cdot P_{\pi^-})P_\omega \\ & - (P \cdot P_{\pi^-})(Q \cdot P_{\pi^+})P_\omega] \\ & \frac{1}{(m_\omega^2 - m_0^2 - im_\omega \Gamma_\omega)} \end{aligned} \quad (76)$$

V. CONCLUSION

Some general formulas for simplifying Lorentz invariant amplitude expressions were developed. These were applied to calculate the Lorentz invariant amplitudes for processes contributing to $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$. These formula are also applicable to $e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-$, $e^+e^- \rightarrow K^+K^-\pi^0\pi^0$, $e^+e^- \rightarrow K^+K^-K^0K^0$, and $e^+e^- \rightarrow K^+K^-K^+K^-$. These formulas may be used to determine the electromagnetic form factors of the ρ meson and properties of the a_1 resonance.

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TABLES

TABLE I. Isobar Model Modes

Mode
$e^+e^- \rightarrow \omega\pi$
$e^+e^- \rightarrow \rho^+\rho^-$
$e^+e^- \rightarrow a_1\pi$
$e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$